

D-Brane Probes and Mirror Symmetry

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Abstract

We study the effect of mirror symmetry for $K3$ surfaces on D-brane probe physics. The case of elliptically fibered $K3$ surfaces is considered in detail. In many cases, mirror can transform a singular fiber of Kodaira's type ADE into sets of singular fibers of type I_1 (II) with equal total Euler number, but vanishing contribution to the Picard number of the mirror surface. This provides a geometric model of quantum splitting phenomena. Mirror for three dimensional gauge theories, interchanging Fayet-Iliopoulos and mass terms, is also briefly discussed.

1 Introduction.

In this note we study mirror symmetry for $K3$ surfaces [1]. We consider in detail the case of $K3$ surfaces which are elliptic fibrations. The physical implications of mirror transformations are studied using two and three D-brane probes [2, 3, 4], depending if we work M or F theory compactifications on $K3$. In the context of probe physics, we observe the following implications of mirror transformations. Given a 3D-probe [2] with Seiberg-Witten moduli [5] an elliptically fibered $K3$ surface, we observe that if we start with the $N=4$ classical solution, characterized by constant τ , the mirror $K3$ elliptically fibered surface, describe the quantum corrected $N=2$ Seiberg-Witten solution with the mass parameters naturally appearing as moduli of the mirror $K3$ surface. In this context the splitting phenomena of a classical singularity into quantum singularities of the moduli appears as a result of mirror transformations on Shioda-Tate formula for elliptically fibered $K3$ surfaces with trivial group of sections [6, 7]. The contribution of the “classical” singularity to the Picard lattice become part of moduli of the mirror surface where new type of singularities with no contribution to Picard should be introduced. Using the geometric mirror map [8, 9] we observe that quantum corrections to the Coulomb branch of the probe dynamics can be mapped into “internal” instanton effects on the special lagrangian D-brane submanifold used to define the geometric mirror. To conclude we make some observations on the mirror pairs in 3D recently discovered by Intriligator and Seiberg [10, 11, 12, 13], where the interchange between mass terms and Fayet-Iliopoulos terms become part of the mirror map on elliptically fibered $K3$ surfaces. An extended version of the work presented in this note is under preparation [14].

2 $K3$ surfaces

A $K3$ surface is characterized by the Hodge diamond

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \quad (1)$$

The space $H_2(X; \mathbf{Z}) \simeq \mathbf{Z}^{22}$ is a self dual lattice of signature $(3, 19)$,

$$\Gamma_{3,19} = E_8 \perp E_8 \perp \mathcal{U} \perp \mathcal{U} \perp \mathcal{U}, \quad (2)$$

where \mathcal{U} , the hyperbolic plane, is the lattice \mathbf{Z}^2 with $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and E_8 is the root lattice of E_8 with reversed sign,

$$E_8 \equiv \begin{pmatrix} -2 & & & & & & & \\ & -2 & 1 & & & & & \\ & 1 & -2 & 1 & & & & \\ & & 1 & -2 & 1 & & & \\ 1 & & & 1 & -2 & 1 & & \\ & & & & 1 & -2 & 1 & \\ & & & & & 1 & -2 & 1 \\ & & & & & & 1 & -2 \end{pmatrix}. \quad (3)$$

A *marking* of the $K3$ surface is defined by an isomorphism of lattices,

$$\phi : H_2(X; \mathbf{Z}) \longrightarrow \Gamma_{3,19}. \quad (4)$$

Given a complex structure we get the Hodge decomposition

$$H^2(X; \mathbf{C}) = H^{0,2}(X) \oplus H^{1,1}(X) \oplus H^{2,0}(X). \quad (5)$$

Let Ω be a holomorphic 2-form. The periods of the $K3$ surface are defined by

$$\overline{\omega}_i = \int_{e_i} \Omega, \quad (6)$$

with e_i a basis of $H_2(X; \mathbf{Z})$. Now we define the Picard lattice $\text{Pic}(X)$ by

$$\text{Pic}(X) \equiv H_{1,1}(X) \cap H_2(X; \mathbf{Z}). \quad (7)$$

$\text{Pic}(X)$ defines a sublattice $\Gamma_{1,t}$ of $\Gamma_{3,19}$. The rank of this lattice is given by

$$\rho(X) = 1 + t. \quad (8)$$

It is clear from (7) that $\rho(X) \leq 20$. The holomorphic 2-form $\Omega \in H_2(X; \mathbf{Z})$ can be associated with a spacelike 2-plane in $\mathbf{R}^{3,19}$:

$$\Omega = x + iy, \quad (9)$$

with $x, y \in H_2(X; \mathbf{C}) \simeq \mathbf{R}^{3,19}$. From (7) it follows that the 2-plane Ω is orthogonal to $\text{Pic}(X)$.

The Teichmüller space of complex structures [15] is given by the grassmannian manifold of 2-spacelike planes in $\mathbf{R}^{3,19}$:

$$\mathcal{T}_C = O(3, 19)/O(2) \times O(1, 19). \quad (10)$$

Modding out by changes of marking we get the moduli space of complex structures

$$\mathcal{M}_C = O(3, 19; \mathbf{Z}) \backslash O(3, 19) / O(2) \times O(1, 19). \quad (11)$$

Generic changes of the complex structure will not preserve $\text{Pic}(X)$. The moduli of complex structures preserving $\text{Pic}(X)$ would be

$$O(\Lambda) \backslash O(2, 19 - t) / O(2) \times O(19 - t), \quad (12)$$

where the *transcendental lattice* Λ is defined by

$$\Lambda = \text{Pic}(X)^\perp, \quad (13)$$

the orthogonal lattice of $\text{Pic}(X)$ in $\Gamma_{3,19}$. Thus, Λ is a lattice of type $(2, 19 - t) = (2, 20 - \rho(X))$.

3 Quantum Cohomology and Mirror Surfaces.

Given X with a Picard Lattice $\text{Pic}(X)$, we can try to find a mirror $K3$ surface Y such that $\text{Pic}(Y)$ is given by the transcendental lattice of X . In order to do that, we need to extend the notion of $\text{Pic}(X)$ to that of *quantum* Picard $\Upsilon(X)$ [15]. Let us define

$$\Upsilon(X) \equiv \text{Pic}(X) \perp \mathcal{U}. \quad (14)$$

Thus, $\Upsilon(X)$ is a lattice of type $(2, t + 1) = (2, \rho)$. In this sense, we can define the mirror $K3$ surface Y through the condition

$$\Upsilon(Y) = \Lambda(X), \quad (15)$$

with $\Lambda(X)$ the transcendental lattice of X . Notice that if $\text{Pic}(X)$ is of type $(1, t)$, then $\text{Pic}(Y)$ is of type $(1, 18 - t)$:

$$\text{rank Pic}(X) + \text{rank Pic}(Y) = 20. \quad (16)$$

The Teichmüller space of complex structures of the mirror surface preserving $\text{Pic}(Y)$ is given by

$$\mathcal{T}_C(Y) = O(2, t + 1) / O(2) \times O(t + 1). \quad (17)$$

The physical origin of mirror symmetry comes from the fact that

$$\mathcal{T}_\sigma = \mathcal{T}_C(X) \otimes \mathcal{T}_C(Y) \quad (18)$$

is the Teichmüller space for the σ -model defined on $K3$.

The mirror symmetry interchanging X and Y is now part of the modular group for the σ -model moduli,

$$\mathcal{M}_\sigma = \mathcal{T}_\sigma / \mathcal{M} \times \mathcal{M}_X \times \mathcal{M}_Y = O(4, 20; \mathbf{Z}) \backslash O(4, 20) / O(4) \times O(20), \quad (19)$$

where M in (19) symbolically represents the mirror symmetry transformation.

Some examples

Notice that (19) is the Narain lattice for the heterotic string compactified on T^4 to 6 dimensions. Moreover, (19) is also the moduli of type II_A string compactified on $K3$. Recall type II_A string contains RR sector with a one form and a three form. However, they do not contribute to the moduli, since $H_1 = H_3 = 0$ for $K3$ surfaces.

We can now consider a case with $\text{Pic}(X) = \Gamma_{1,1}$. The moduli space (12) is given by

$$O(2, 18; \mathbf{Z}) \backslash O(2, 18) / O(2) \times O(18), \quad (20)$$

which is the Narain lattice for heterotic string on T^2 . This is a result known as duality between F -theory [16] on $K3$, with $\text{Pic}(K3) = \Gamma_{1,1}$ and heterotic string on T^2 [17].

As another example, we can consider M -theory on $K3$: we must take the moduli of M -theory on $K3$ to be

$$O(3, 19; \mathbf{Z}) \backslash O(3, 19) / O(3) \times O(19), \quad (21)$$

that corresponds to heterotic string on T^3 .

In a certain sense we observe that M and F theories are different ways of mapping $K3$ moduli into heterotic moduli.

4 Polarized $K3$ Surfaces.

Following Dolgachev [18], we define an M -polarized $K3$ surface as a pair (X, j) , with j a lattice embedding,

$$j : M \longrightarrow \text{Pic}(X), \quad (22)$$

such that

$$\phi^{-1}(M) \in \text{Pic}(X). \quad (23)$$

We will take M to be a lattice of type $(1, t)$. As before, the moduli of complex structures of M -polarized $K3$ surfaces would be given by (12), with $t + 1$ the rank of M . We will take

$$\text{rank } M = \text{rank } \text{Pic}(X). \quad (24)$$

To define the mirror surface we choose an isotropic vector f , with $(f, f) = 0$, in M^\perp , the orthogonal to M in $\Gamma_{3,19}$. Defining

$$M^* = f^\perp / f, \quad (25)$$

the mirror $K3$ surface is the M^* -polarized $K3$ surface. Notice that if M is of type $(1, t)$, then M^* is of type $(1, 18 - t)$, as required by mirror. As an example, consider

$$M = \langle 2n \rangle, \quad (26)$$

the lattice defined by $e \cdot e = 2n$. The orthogonal M^\perp is given by

$$M^\perp = \mathcal{U} \perp \mathcal{U} \perp E_8 \perp E_8 \perp \langle -2n \rangle, \quad (27)$$

and M^* ,

$$M^* = \mathcal{U} \perp E_8 \perp E_8 \perp \langle -2n \rangle. \quad (28)$$

Notice that choosing f in this construction is equivalent to finding a “classical” Picard sublattice in M^\perp . The example (26) corresponds to $t = 0$, and M^* in (28) is of rank = 19.

5 Elliptic Fibrations.

A $K3$ surface X is an elliptic fibration if

$$\Pi : X \longrightarrow \mathbf{P}^1 \quad (29)$$

with $\Pi^{-1}(z)$ an elliptic curve. The basic information on an elliptic fibration is given by its degenerate fibers. They were classified by Kodaira (see Table 1) [19].

Denoting F_v the set of singular fibers, we have

$$24 = \sum e(F_v), \quad (30)$$

with $e(F_v)$ the Euler number of the singular fibers. An equivalent condition to (30) is given by the adjunction formula [20],

$$K_x = \Pi^*(K_{\mathbf{P}^1} + \sum a_i P_i), \quad (31)$$

with P_i the points on the base space where the fiber becomes singular, and a_i given in Table 1.

From Shioda-Tate lemma the following formula can be derived:

$$\rho(X) = 2 + \sum_v \sigma(F_v) + \text{rank } \Phi, \quad (32)$$

where $\rho(X)$ is the Picard number of X , Φ the group of sections and $\sigma(F_v) + 1$ the number of components of the singular fiber F_v . In Kodaira’s classification this number is given by the number of points of the affine Dynkin diagram (see Table 1).

Kodaira	Dynkin	e	σ	a
I_n	A_{n-1}	n	$n-1$	$n/12$
I_n^*	\bar{D}_{4+n}	$n+6$	$n+4$	$1/2 + n/12$
II		2	0	$1/6$
III		3	1	$1/4$
IV		4	2	$1/3$
II^*	\bar{E}_8	10	8	$5/6$
III^*	\bar{E}_7	9	7	$3/4$
IV^*	\bar{E}_6	8	6	$2/3$

The meaning of the Shioda-Tate lemma is the following. Given the elliptic fibration, we can define $\text{Pic}'(X)$ as the Picard sublattice containing a fiber F and a section S satisfying

$$\begin{aligned} F \cdot S &= 1, \\ S \cdot S &= -2, \\ F \cdot F &= 0, \end{aligned} \tag{33}$$

and the sum of lattices of type ADE spanned by the irreducible components of fibers not intersecting the section S . The rank of $\text{Pic}'(X)$ is given by

$$\rho'(X) = 2 + \sum_v \sigma(F_v). \tag{34}$$

Thus, equation (32) becomes equivalent to the isomorphism between $\text{Pic}(X)/\text{Pic}'(X)$ and the group of sections of the fibration.

We can now consider some examples. Let us consider $M = \langle +2 \rangle$; the mirror manifold is defined by¹

$$M^* = \mathcal{U} \perp E_8 \perp E_8 \perp \langle -2 \rangle. \tag{35}$$

This lattice is of rank = 19. Let us now consider this lattice as $\text{Pic}'(X)$ with a trivial group of sections. Using Table 1 and equation (32), we can define the mirror as an elliptic fibration with two fibers of type E_8 and a fiber with $\sigma = 1$, i.e., either of type I_2 or III . In order to saturate (30), we need to add singular fibers of type I_1 , i.e., with $\sigma = 0$.

Using the same type of arguments we can consider some pairs of mirror $K3$ surfaces which are both elliptically fibered. Let us consider for instance a case with $\rho(X) = 2$, and 24 singular fibers of type I_1 . In this case, the Picard lattice $\Gamma_{1,1}$ would be interpreted as

¹As shown by Dolgachev [18], (35) defines the mirror [21, 22] to the toric manifold defined by $(1, 1, 1, 3)$, $x_1^6 + x_2^6 x_3^6 + x_4^2 = 0$. This $K3$ surface is specially interesting, since it is the one appearing in the $K3$ -fibration of the Calabi-Yau space $\mathbf{P}_{\{1,1,2,2,6\}}^{12}$, used to obtain Seiberg-Witten moduli from heterotic-type II dual pairs [23, 24].

generated by a section S and a fiber F satisfying relations (33). The mirror to this $K3$ surface would be given by

$$M = \mathcal{U} \perp E_8 \perp E_8, \quad (36)$$

with Picard number 18. This corresponds to an elliptic fibration with two E_8 singularities and extra I_1 's not contributing in (32) to the Picard number $\rho(X)$. This is the elliptic fibration used in F-theory to define the type II_B dual of heterotic string on T^2 , with unbroken $E_8 \times E_8$ symmetry, i.e., no Wilson lines [17].

6 Mirror Symmetry and Sen's Orientifold Model.

In [2] Sen has considered an elliptically fibered $K3$ consisting of four D_4 singularities. This corresponds to constant τ , since for D_n singularities the monodromy is given by

$$\begin{pmatrix} -1 & -n+4 \\ 0 & -1 \end{pmatrix}. \quad (37)$$

From Shioda-Tate lemma and assuming a trivial group of sections we get

$$\rho = 18 \quad (38)$$

as the Picard number. The situation corresponding to a D_4 singularity can be described in terms of a type IIB compactification with four orientifold planes, contributing with charge -4 , and four groups of 7-branes compensating locally the charge of the orientifold [2]. This explains in type IIB language the constant value of τ . Equivalently, in F-theory language each D_4 singularity contributes with six units to the total Euler number. Let us now consider the mirror to the model defined by four D_4 singularities. This is a $K3$ surface with Picard lattice of type $\Gamma_{1,1}$. The dimension of the moduli space for the mirror surface is given by the Picard number (38), i.e., 18. The singular fibers for the mirror surface should be of type I_1 or II in Kodaira's notation. In fact they contribute to the total Euler number, but not to the Picard number of the mirror $K3$ surface, which is equal to two. The 18 moduli of the mirror can be described using four $SU(2)$ $N_f = 4$ SUSY gauge theories, each one describing locally a "quarter" of the base space [2]. For each of these theories we have four moduli corresponding to the masses of the $N_f = 4$ hypermultiplets. The other two common moduli, up to the total of 18, correspond to the complex and Kähler moduli of the heterotic dual on T^2 . The moduli τ_0 in Seiberg-Witten solution for $SU(2)$ with $N_f = 4$ is related to the ratio, fixed by the j -function, of these two moduli.

The interest of Sen's result is in part due to the observation that the F-theory solution contains the quantum corrections, which are at the origin of the orientifold splitting. Our previous analysis in terms of mirror $K3$ surfaces suggest the following general picture: In going to the mirror surface we replace the D_4 singularity by a set of singularities with

total Euler number equal to 6, but not contributing to the Picard number. Symbolically, we can define the mirror action on the singular fibers as follows²:

$$\begin{aligned} M : D_4 &\longrightarrow \oplus_{i=1}^6 I_1^i, \\ M : \sigma(D_4) &\longrightarrow \sigma(\oplus_{i=1}^6 I_1^i) = 0. \end{aligned} \quad (39)$$

Generically, and for trivial group of sections, for any singular fiber S of ADE type, the “mirror” S^* would satisfy

$$\begin{aligned} e(S^*) &= e(S), \\ \sigma(S^*) &= 0. \end{aligned} \quad (40)$$

By this process we pass from the moduli of the original theory described by the D_4 singularities to the 18 dimensional moduli of the mirror manifold. The Seiberg-Witten theory used by Sen, i.e., the quantum moduli of the 3-brane probe world volume field theory, is parametrized by these moduli. In summary we observe that quantum corrections are automatically encoded in the $K3$ mirror surface, i.e., the moduli of the mirror parametrize the quantum deformations. Moreover the splitting phenomena can be also interpreted as a mirror effect that replaces a singularity contributing to the Picard number by a set of singularities with equal Euler number but not contributing, in the Shioda-Tate formula, to the total Picard number of the mirror manifold³.

7 Geometric Mirror and D-Branes

In reference [8] a geometric characterization of mirror manifolds was proposed using toroidal lagrangian submanifolds. In this approach, mirror becomes equivalent to T-duality. Again, we will reduce our analysis to the simpler case of $K3$ surfaces. A special lagrangian submanifold in a $K3$ surface X is a compact complex one dimensional manifold M with an immersion $f : M \rightarrow X$, such that $f^*(\Omega)$ coincides with the induced volume form on M . The moduli of special lagrangian submanifolds is determined by Mc Lean’s theorem. Its dimension is equal to $b_1(M)$. We will take for M the 1-torus T with $b_1(T)=1$. Denoting $M(T, X)$ the moduli of immersions, the space X becomes an elliptic fibration with base space $M(T, X)$. Now, we consider a $U(1)$ flat bundle on T , and define the D-brane moduli space $M_D(T, X)$ containing the moduli of immersions and the moduli of flat $U(1)$ bundles. The space $M_D(T, X)$ fibers on $M(T, X)$. The geometric mirror is the statement that $M_D(T, X)$ is the mirror $K3$ surface⁴. Thus in this approach we pass

²Other possibilities for the quantum splitting (39) are $D_4 \rightarrow \oplus_{i=1}^3 II^i$; $D_4 \rightarrow II \oplus II \oplus I_1 \oplus I_1$; $D_4 \rightarrow II \oplus \oplus_{i=1}^3 I_1^i$.

³The use of mirror symmetry to derive quantum moduli for supersymmetric gauge theories is also considered in reference [25]. It would be interesting to compare both approaches.

⁴It is important to notice the similarity between $M_D(T, X)$ and Donagi-Witten [26] construction of integrable models. For the hyperkähler case, i. e., $K3$ surfaces, we can use to describe the moduli space of

from an elliptic fibration to a mirror surface which is also elliptically fibered. Based on Mukai's results [27], Morrison [9] argues that, at least for $K3$ surfaces, geometric mirror coincides with the standard concept of mirror above described. Geometric mirror provides a different point of view on the dynamical origin of the quantum corrections originating the splitting phenomena. Coming back to Sen's case, if the "mirror" of the D_4 singularity is described by six I_1 singularities and at the same time this mirror elliptic fibration is interpreted as $M_D(T, X)$, then we observe that the dynamics underlying the "splitting" of D_4 into I_1 's is due to instanton effects on the D-brane T , that we should describe as a disc in X winding on one-cycles of T [8]. In this way, "internal manifold" instanton effects allow us to pass from the "classical" τ constant solution to the quantum "mirror" Seiberg-Witten solution.

T-Duality and Probes

The equivalence between geometric mirror and T-duality, can be symbolically represented as follows:

$$\mathcal{M}(\bullet; X^*) = \mathcal{M}_D(T; X), \quad (41)$$

where $\mathcal{M}(\bullet, X^*)$ represents the mirror of a 0-brane on the $K3$ surface X^* , and $\mathcal{M}_D(T; X)$ the D-brane moduli on the mirror manifold⁵. Notice that for $K3$ surfaces mirror is duality in the sense that

$$X^{**} = X. \quad (42)$$

Now we can consider the case of M-theory on $K3$ and consider a 2-brane probe defining a 3D SUSY theory on its worldvolume. The moduli of this 3D SUSY theory is given by $K3$ itself. This 2-brane looks from the $K3$ point of view as a 0-brane, and therefore we can identify the left hand side of (41) with the moduli of the 3D SUSY theory defined on the 2-brane worldvolume [29, 30]. Now, how should we interpret (41) in terms of probe dynamics?

We can try to interpret the right hand side of (41) as the moduli of a 4-brane with worldvolume space $\mathbf{R}^2 \times S^1 \times S^1$, i. e., a 5D gauge theory. In this case, the moduli of this 5D [31] would be given by the base space of the fibration $\mathcal{M}_D(T; K3)$, i. e., the Mc Lean space of deformations of immersions of T in $K3$. When we compactify on T to three dimensions we get the moduli of the 3D theory of the 2-brane probe on the *mirror surface*.

Moreover, after compactification from 5D to 3D on T^2 the theory on the 2-brane probe possesses $N=4$ matter content. Again, we find the same picture as in Sen's example. The case with $N=4$ goes to the deformed $N=2$ in passing to the mirror surface.

Another specially interesting case of geometric mirror is the one of Calabi-Yau four-folds, where the special lagrangian manifold is a complex four torus. The results of

immersions of T in X the holomorphic one forms on T . In this sense, and interpreting T as the reference surface E_τ in [26], the Higgs field ϕ on E_τ can be interpreted as parametrizing the different immersions on $K3$.

⁵The map from 2-cycles into 0-cycles is part of the Fourier-Mukai transform [28] for $K3$

reference [32] concerning Seiberg's duality [33] for $N=1$ gauge theories can be interpreted as a consequence of mirror symmetry on the fourfold, i. e., T -duality transformations on the special toroidal lagrangian. This example indicates that F-theory compactifications on mirror manifolds produce dual field theories on the worldvolume probe.

8 Mirror Symmetry in 3D

For 3D gauge theories a type of mirror has been recently discovered by Intriligator and Seiberg [10] using the D-brane probe philosophy. Mirror symmetry in this case interchanges the Coulomb and Higgs branches. If the Coulomb branch is a $K3$ surface with ADE singularities, the Higgs branch is defined as the corresponding moduli of ADE instantons. Notice that the $K3$ is elliptically fibered and that in the D-probe philosophy singular fibers define the global symmetries for the 2-brane probe worldvolume physics. In more concrete terms, mirror symmetry discussed in [10] interchanges masses, on which the Coulomb branch geometry depends quantum mechanically, and Fayet-Iliopoulos D-terms, which can only affect the metric of the Higgs branch. This is symmetry consistent with our approach in previous sections to mirror symmetry in the following sense: the number of Fayet-Iliopoulos terms is precisely given by the value of σ for the singularity (see Table 1). This is the contribution to the Picard lattice in Shioda-Tate formula of the corresponding singularity. Generically, as we pass to the mirror $K3$ surface we convert this contribution to Picard into moduli parameters of the mirror surface. These moduli parameters, as was described in Sen's example, are interpreted as mass terms on which the metric of the Coulomb branch depends. In this sense, the interchange between Fayet-Iliopoulos and mass terms reflects the transformation, by mirror symmetry, of the Picard number of the singular fibers into moduli of the mirror surface.

Notice that for Kronheimer ADE gauge theories, the number of mass parameters for A_{n-1} theories is one and zero for D, E_6, E_7, E_8 theories. In our approach, the mass term for A_{n-1} theories comes from the fact that for I_n singularities $e(I_n) = \sigma(I_n) + 1$, while for D, E_6, E_7, E_8 we have $e(\) = \sigma(\) + 2$.

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